(1) Problem

a) $u_{xx} + 2u_{xy} + 17u_{yy} = 0$

canonical form is $\downarrow$ 2pts

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0$$

~~$a_{11}\, a_{22}$~~ $\Rightarrow$ $a_{11} = 1 \quad a_{12} = 1 \quad a_{22} = 17$

2pts

$$\left.\begin{array}{l} a_{11}a_{22} = 17 \\ a_{22}^2 = 1 \end{array}\right\} \Rightarrow \text{ since } a_{11}a_{22} > a_{22}^2$$

$$D = 17 - 1 > 0$$

the PDE is <u>elliptic</u>; and of the form $u_{\xi\xi} + u_{\eta\eta} = 0$. } 4pts to conclusion

b) $x^2 + 2xy + 17y^2 = P(x,y)$

then $(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^2 + 16y^2$ is the

quadratic polynomial associated with the coordinate transformation.

Let $\xi = x + y$ $\quad \eta = 4y$

this turns the equation into

$$u_{\xi\xi} + u_{\eta\eta} = 0$$

2 pts

b) This is $\boxed{\text{elliptical}}$ + the solution attains its maximum on the boundary of the unit disk by the $\boxed{\text{harmonic maximum principle}}$ 3pts

c) $$u(\eta, \xi) = \int_D \frac{h(\eta)\left(1 - \left(\eta^2 + \xi^2\right)\right)}{2\pi \left|(\eta, \xi) - (\tilde{\eta}, \tilde{\xi})\right|^2} \, d\tilde{\eta} \, d\tilde{\xi}$$

This is the Poisson integral formula. 3pts

Making the change of variable of variable

$$(\tilde{\eta}, \tilde{\xi}) \mapsto (\tilde{x}, \tilde{y})$$

$$\underbrace{J = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} = 4}_{}$$

3pts to change of variables

$$u(x,y) = \int_{\tilde{D}} \frac{4\ln(y/4)\left(1-\left(\frac{y}{4}\right)^2+(x+y)^2\right)}{2\pi\left(\left(\frac{y}{4}, x+y\right)-\left(\frac{\tilde{y}}{4}, \tilde{x}+\tilde{y}\right)\right)^2}\, d\tilde{x}\, d\tilde{y}$$

is the result in the plane

1 pt to final solution

Problem 2a

Note that the function

$$v = \frac{-k}{2c^2}(x^2 - 2lx) \qquad 5\text{ pts}$$

satisfies the boundary conditions and the PDE

Therefore $w = u - v$ satisfies

$$w_{tt} = c^2 w_{xx} \qquad 3\text{ pts}$$

$$w(0,t) = w_x(l,t) = 0$$

$$w(x,0) = \frac{k}{2c^2}(x^2 - 2lx) \quad w_t(x,0) = 0$$

The eigenfunctions of the homogeneous problem are $\sin(\beta_n x)$ $\beta_n = \frac{(n+\frac{1}{2})\pi}{l}$

for $n \geq 0$ we may write:

$$w(x,t) = \sum_{n=1}^{\infty} \sin(\beta_n x)\left[A_n \cos(\beta_n x c t) + B_n \sin(\beta_n c t)\right]$$

2pts

the initial conditions

$$w(x,0) = \frac{k}{2c^2}(x^2 - 2lx)$$

implies

2pts

$$A_n = \frac{2}{l}\int_0^l \frac{k}{2c^2}(x^2 - 2lx)\sin(\beta_n x)\,dx$$

$$= \frac{-2k}{lc^2\beta_n^3}$$

$w_t(x,0) = V$ implies

$$B_n = \frac{2}{l}\int_0^l \frac{V}{c\beta_n}\sin(\beta_n x)\,dx = \frac{2V}{lc\beta_n^2}$$

2pts

Thus

$$u(x,t) = \frac{-k}{2c^2}(x^2 - 2lx)$$

$$+ \sum_{n=1}^{\infty} \sin(\beta_n x)\left[\frac{-2k}{lc^2\beta_n^3}\cos(\beta_n ct) + \frac{2V}{lc\beta_n^2}\sin(\beta_n ct)\right]$$

1 pt to final solution

Problem 26

We did in lecture that the wave equation when Fourier transformed has solutions $\cos(ckt)$ and $\frac{\sin(ckt)}{ck}$ 5 pts each.

where k is the dual variable.

then 2 pts

$$\mathcal{F}^{*}(Ve^{-|x|}) = V\mathcal{F}(e^{-|x|})$$

$$= V\,\mathcal{F}(e^{-|x|}) = \frac{2V}{1+k^2}$$

This is according to the table of Fourier transforms students are allowed to use.

The final solution is 3 pts

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(ckt)}{ck}\left(\frac{2V}{1+k^2}\right) e^{+ik\cdot x}\, dk$$

$$= u(x,t)$$

Problem 3

The solution to

$$w_t + bt^2 w = 0 \qquad 5\text{pts}$$

is $e^{-bt^3/3}$

Then $u_t = \frac{\partial}{dt}(w\, v(t,x)) = \frac{d}{dt}\left(e^{-bt^3/3} v\right)$

$$= e^{-bt^3/3}\left(-bt^2 v + v_t\right)$$

5pts

also $u_{xx} = e^{-bt^3/3} v_{xx}$.

Substituting into the equation

$u_t - k u_{xx} + bt^2 u = 0$ gives

$$v_t - k v_{xx} = 0 \qquad 5\text{pts}$$

When $t=0$ $u=v$ so the initial data is unchanged.

$$u(x,t) = \frac{e^{-bt^3/3}}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy$$

5pts to solution

Because $G''(x)=0$ for $x \neq x_0$

$$G(x) = \begin{cases} a_1 x + b_1 & 0 < x < x_0 \\ a_2 x + b_2 & x_0 < x < l. \end{cases}$$

Now $G(0)=0$ implies $b_1 = 0$ $G(l) = 0$ implies $b_2 = -a_2 l$ so that

$$G(x) = \begin{cases} a_1 x & 0 < x < x_0 \\ a_2 x - a_2 l & x_0 < x < l. \end{cases}$$ 5pts

Then $G(x)$ is continuous at x_0 (Because $G(x) + \frac{1}{2}|x - x_0|$ is continuous at x_0). Therefore $a_1 x_0 = a_2 x_0 - a_2 l$

which implies $a_1 = a_2 \left(1 - \frac{l}{x_0}\right)$.

Lastly the fact $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0 means the derivative of $G(x) + \frac{1}{2}|x - x_0|$

to be continuous at x_0 5pts

$$x < x_0 \Rightarrow \quad G(x) + \frac{1}{2}|x - x_0| = a_1 x + \frac{1}{2}(x_0 - x)$$

$$\Rightarrow \frac{d}{dx}\left(G + \frac{1}{2}|x - x_0|\right) = a_1 - \frac{1}{2}$$

$$x > x_0 \Rightarrow \quad G(x) + \frac{1}{2}|x - x_0| = a_2 x - a_2 \ell + \frac{1}{2}(x - x_0)$$

$$\Rightarrow \frac{d}{dx}\left(G + \frac{1}{2}|x - x_0|\right) = a_2 + \frac{1}{2}$$

— 5pts

We need $a_2 + \frac{1}{2} = a_1 - \frac{1}{2}$

Using the fact $a_1 = a_2\left(1 - \frac{\ell}{x_0}\right)$

we see that $a_2 + \frac{1}{2} = a_2\left(1 - \frac{\ell}{x_0}\right) - \frac{1}{2}$

which implies $a_2 = -x_0/\ell \quad a_1 = -x_0/\ell + 1$

We therefore conclude that

$$G(x) = \begin{cases} \left(-\frac{x_0}{\ell} + 1\right)x & 0 < x < x_0 \\ -\frac{x_0}{\ell}(x - \ell) & x_0 < x < \ell. \end{cases}$$

5pts.